International Journal of Theoretical Physics, Vol. 38, No. 1, 1999

Lie–Nambu and Beyond

Marek Czachor[1]

Received August 7, 1998

Linear quantum mechanics can be regarded as a particular example of a nonlinear Nambu-type theory. Some elements of this approach are presented.

1. INTRODUCTION

At the moment there is no single experimental result suggesting that states of a quantum system can evolve in a fundamentally nonlinear way. On the other hand, all impossibility theorems stating that such a nonlinearity is in principle impossible have not survived a detailed analysis. It is therefore possible that the status of quantum linearity is similar to that of geometrical linearity before the invention of general relativity.

The multiple-bracket dynamics described in this paper arose from a search for a consistent embedding of linear quantum mechanics into a more general theory where the assumption of linearity could be dropped. The formalism is essentially based on density matrices and not on wave functions. A density matrix plays here the role of a fundamental field and should not be regarded as a mixture of classical and quantum probabilities. A departure point for the discussed generalization is the observation that density matrices of ordinary quantum mechanics satisfy an equation of a Lie–Nambu type.

The layout of the paper is as follows. Section 2 relates the work to the earlier efforts by Nambu (1973) and Białynicki-Birula and Morrison (1991). Some formal tools are introduced in Section 3–5, In Section 6 a $(2n + 1)$-bracket is introduced and some of its general properties are proved. The bracket differs from the so-called generalized Nambu, generalized Poisson, or generalized Nambu–Poisson brackets discussed in the literature (Bayen

[1] Katedra Fizyki Teoretycznej i Metod Matematycznych, Politechnika Gdańska, 80-952 Gdańsk, Poland; e-mail: mczachor@sunrise.pg.gda.pl.

0020-7748/99/0100-047 5$16.00/0

and Flato, 1975; Cohen, 1975; Takhtajan, 1994; Gautheron, 1996; Chatterjee and Takhtajan, 1996; de Azcárraga *et al.*, 1996; Ditto and Flato, 1997; Ibánez *et al.*, 1997; Ditto *et al.*, 1997; Kanatchikov, 1997). The particular case of a 3-bracket dynamics is discussed in Section 7, where the notion of a Lie–Nambu duality is introduced and properties of solutions of the 3-bracket equations are discussed. The 5-bracket dynamics is briefly discussed in Section 8. Section 9 is devoted to the question of N-particle extensions of a nonlinear Lie–Poisson dynamics. The notion of complete separability is discussed and examples of completely separable equations are given. The results of this section contradict the popular belief that all nonlinear extensions of quantum mechanics lead to faster-than-light phenomena. Also the question of complete positivity of solutions is briefly discussed in this section. Section 10 is devoted to the problem of separability of the dual Poisson dynamics and it is shown that a surprising nonlocal phenomenon occurs. In Section 11 a possible link between our formalism and the problem of quantization of classical Nambu dynamics is discussed.

2. LIE–POISSON BRACKET AS A LIE–NAMBU BRACKET

The origin of this work goes back to two papers where, in completely different contexts, a notion of a triple bracket was introduced.

2.1. Nambu (1973): Euler Equations

The Euler equations for a rotating rigid body are

$$J_k = \varepsilon_{kbc}(J_b/I_b)J_c \tag{1}$$

$$= J_a\varepsilon^a{}_{bc}\frac{\partial J_k}{\partial J_b}\frac{\partial H}{\partial J_c} = \{J_k, H\} \tag{2}$$

$$= \varepsilon_{abc}\frac{\partial J_k}{\partial J_a}\frac{\partial H}{\partial J_b}\frac{\partial S}{\partial J_c} = \{J_k, H, S\} \tag{3}$$

Here $\mathbf{J}$ is the angular momentum, I_k the component of the moment of inertia, $H = J_1^2/2I_1 + J_2^2/2I_2 + J_3^2/2I_3$ is the rotational energy, and $S = \frac{1}{2}\mathbf{J}^2$. The totally antisymmetric tensor ε_{abc} can be regarded either as a 3-dimensional volume form or as structure constants of $so(3)$. The Lie algebra $so(3)$ enters the equations also via S since $\mathbf{J}^2$ is a second-order Casimir invariant of this algebra. The form (2) defines a Poisson bracket. The triple bracket defined by (3) is nowadays called the Nambu bracket and was introduced in Nambu (1973).

The Poisson bracket (2) is a particular case of the so-called Lie–Poisson bracket, which differs from (2) by the presence of structure constants c^a_{bc} of

some Lie algebra instead of ε^a_{bc} characteristic of $so(3)$. It is natural to think of the Nambu bracket as a particular case of

$$\{A, B, S\} = c_{abc} \frac{\partial A}{\partial x_a} \frac{\partial B}{\partial x_b} \frac{\partial S}{\partial x_c} \tag{4}$$

where $S = \frac{1}{2} g^{ab} x_a x_b$ is a second-order Casimir invariant of an appropriate Lie algebra. Such brackets could be called Lie–Nambu brackets and are quite natural in the context of generalizations of a Lie–Poisson dynamics. It is surprising that this kind of generalization of classical Hamiltonian dynamics has not been considered so far in the theory of classical dynamical systems (Ratiu, n.d.). One of the reasons seems to be the fact that for general Lie algebras the bracket does not satisfy the so-called fundamental identity (Chatterjee and Takhtajan, 1996). We shall return to this question in Section 7.

A generalization of (3) which has been extensively investigated in the literature under the name of a "generalized Nambu dynamics" goes in another direction (Bayen and Flato, 1975; Takhtajan, 1994; Gautheron, 1996; Chatterjee and Takhtajan, 1996). One treats the ε not as structure constants, but as a volume form. From this perspective it is natural to consider

$$\{A_1, \ldots, A_n = \varepsilon_{a_1 \ldots a_n} \frac{\partial A_1}{\partial x_{a_1}} \ldots \frac{\partial A_n}{\partial x_{a_n}} \tag{5}$$

The parameter n is a dimension of the state space. It is not clear how to extend this type of description to infinite-dimensional spaces.

2.2. Białynicki-Birula and Morrison (1991): Liouville–von Neumann equation

The observation that the Liouville–von Neumann equation for a Wigner function can be written as a Lie–Nambu equation with nontrivial structure constants is due to Białynicki-Birula and Morrison (1991). Below, instead of the Wigner function, which is defined in terms of position–momentum coordinates, we shall stick to the more symmetric position–position representation. This will lead to a specific form of structure constants whose symmetry properties will be essential for further generalizations (Czachor, 1997a; Czachor and Kuna, 1997a).

The density matrix in position representation is denoted by $\rho(a, a') =: \rho_a$, where we use a and a' instead of more typical x and x', and the lower composite index is introduced for brievity. The kinetic energy is represented by the kernel

$$\int dy\ K(a, y)\rho(y, a') = \frac{-\Delta_a}{2m} \rho(a, a') \tag{6}$$

and the Hamiltonian operator by

$$H(a', a) = K(a', a) + V(a)\delta(a - a') \tag{7}$$

It is easy to check that the equation

$$\begin{aligned} i\partial_t \rho_a = & \int db\ db'\ dc\ dc' \\ & \times \underbrace{(\delta(a - b')\delta(b - c')\delta(c - a') - \delta(a - c')\delta(b - a')\delta(c - b'))}_{\Omega_{abc}} \\ & \times \underbrace{H(b', b)}_{H^b}\underbrace{\rho(c', c)}_{\rho^c} \end{aligned} \tag{8}$$

$$= \Omega_{abc} H^b \rho^c \tag{9}$$

is equivalent to the Liouville–von Neumann one. The form (9) simultaneously illustrates the use of composite indices and the summation convention. Notice that the composite indices are in their lower or upper position, and the transition between the two is given by the metric tensor g^{ab} working as follows:

$$g^{ab}\rho_b = \int db\ db'\ \delta(a - b')\delta(b - a')\rho(b, b') = \rho(a', a) = \rho^a \tag{10}$$

So if $\rho_a = \rho(a, a')$, then $\rho^a = \rho(a', a)$. Although the latter formula may seem somewhat artificial and was not used by Białynicki-Birula and Morrison, it will prove extremely useful when we arrive at various generalizations. The distributions Ω_{abc} are structure constants of an infinite-dimensional Lie algebra, which can be checked by raising a with the help of g^{ab} and verifying the standard properties. One should be aware of the fact that g^{ab} is *not* the Cartan–Killing metric (which does not exist in this case). Writing $g^{ab}\rho_a\rho_b = \mathrm{Tr}(\hat{\rho}^2) =: C_2 =: 2S$, one recognizes that g^{ab} is a kernel form of the Hilbert–Schmidt metric. Let now $H(\hat{\rho}) = \mathrm{Tr}\,\hat{H}\hat{\rho}$. Taking into account that $H^a = \delta H/\delta\rho_a$ and $\rho^a = \delta S/\delta\rho_a$ ($\delta/\delta\rho_a$ is a functional derivative), we can write the Liouville–von Neumann equation in the Białynicki-Birula–Morrison form as

$$i\dot{\rho}_k = \rho_a \Omega^a_{bc} \frac{\delta\rho_k}{\delta\rho_b}\frac{\delta H}{\delta\rho_c} = \{\rho_k, H\} \tag{11}$$

$$= \Omega_{abc}\frac{\delta\rho_k}{\delta\rho_a}\frac{\delta H}{\delta\rho_b}\frac{\delta S}{\delta\rho_c} = \{\rho_k, H, S\} \tag{12}$$

Although the relationship of (11) and (12) to (2) and (3), is obvious, it requires a few comments. First of all, the equations are Lie–Poisson and *Lie*–Nambu and not "generalized Nambu" in the sense of the previous subsec-

tion. The Lie algebra and the space of states are both infinite dimensional. H is an average energy and $C_2 = 2S$ is a Casimir invariant called an "entropy." [Actually, it *is*, up to constants, a 2-entropy of Daróczy (1970) or Tsallis (1988), and is closely related to Rényi's α-entropies (Rényi, 1960, 1961).]

3. DIGRESSION ON PURE STATES

We need one more formal prerequisite before we go further. The Liouville–von Neumann equation for mixed states has its roots in the pure-state Schrödinger equation. It turns out that the same is true of the structure constants Ω_{abc}. Let us switch now one level higher and instead of speaking about a nonrelativistic, spin-0 Schrödinger equation consider general Hamilton equations on a separable Hilbert space:

$$i\omega^{\alpha\alpha'}\,\dot{\psi}_\alpha = \frac{\delta H}{\delta \bar{\psi}_{\alpha'}}, \quad -i\omega^{\alpha\alpha'}\overline{\psi}_{\alpha'} = \frac{\delta H}{\delta \psi_\alpha} \tag{13}$$

If $\psi_\alpha = \psi(a)$, $\omega^{\alpha\alpha'} = \delta(a - a')$, $H = \langle\psi|\hat{H}|\psi\rangle$, and an obvious summation/integration convention is applied, (13) is equivalent to the Schrödinger equation (Chernoff and Marsden, 1974). $\omega^{\alpha\alpha'}$ is a symplectic form in the complex coordinates $\psi = q + ip$. The explicit form of $\omega^{\alpha\alpha'}$ varies from representation to representation and is different for, say, the Dirac equation, or a nonrelativistic particle with spin. It is important, however, that the form of the Hamilton equations (13) is always the same [although the dot at its LHS may have different meanings as well; cf. Czachor (1997a), Czachor and Kuna (1997a)] and that $\omega^{\alpha\alpha'}\,\psi_\alpha\bar{\phi}_{\alpha'} = \langle\phi|\psi\rangle$. Equations (13) can be written in a form involving a Poisson tensor

$$i\dot{\psi}_\alpha = I_{\alpha\alpha'}\frac{\delta H}{\delta \bar{\psi}_{\alpha'}}, \qquad -i\overline{\psi}_{\alpha'} = I_{\alpha\alpha'}\frac{\delta H}{\delta \psi_\alpha} \tag{14}$$

A pure-state density matrix is given by $\rho_a = \rho_{\alpha\alpha'} = \psi_\alpha\bar{\psi}_{\alpha'}$ and $\omega^{\alpha\alpha'}\,\rho_{\alpha\alpha'} = \omega^a\,\rho_a = \mathrm{Tr}\,\hat{\rho}$. A pure-state Poisson bracket corresponding to (14) and its complex conjugated equation is

$$\{A, B\} = I_a\frac{\delta A}{\delta \psi_\alpha}\frac{\delta B}{\delta \bar{\psi}_{\alpha'}} - (A \leftrightarrow B) \tag{15}$$

$$= \rho_a\,\Omega^a_{bc}\frac{\delta A}{\delta \rho_b}\frac{\delta B}{\delta \rho_c} \tag{16}$$

which holds for all functions $A(\rho_a) = A(\psi_\alpha \bar{\psi}_{\alpha'})$ and $B(\rho_a) = B(\psi_\alpha \bar{\psi}_{\alpha'})$. The structure constants are

$$\Omega^{a}_{bc} = \delta^{\alpha'}_{\beta'}\delta^{\alpha}_{\gamma} I_{\beta\gamma'} - \delta^{\alpha'}_{\gamma'}\delta^{\alpha}_{\beta} I_{\gamma\beta'} \tag{17}$$

$$\Omega_{abc} = I_{\alpha\beta'} I_{\beta\gamma'} I_{\gamma\alpha'} - I_{\alpha\gamma'} I_{\beta\alpha'} I_{\gamma\beta'} \tag{18}$$

$$\Omega^{abc} = -\omega^{\alpha\beta'}\omega^{\beta\gamma'}\omega^{\gamma\alpha'} + \omega^{\alpha\gamma'}\omega^{\beta\alpha'}\omega^{\gamma\beta'} \tag{19}$$

where the deltas are defined by

$$I_{\alpha\beta'}\omega^{\alpha\alpha'} = \delta^{\alpha'}_{\beta'} \tag{20}$$

$$I_{\beta\alpha'}\omega^{\alpha\alpha'} = \delta^{\alpha}_{\beta} \tag{21}$$

The metric tensor that raises and lowers the composite indices is given by $g^{ab} = \omega^{\alpha\beta'}\omega^{\beta\alpha'}$ and $g_{ab} = I_{\alpha\beta'} I_{\beta\alpha'}$. The reader may check that we obtain the Białynicki-Birula–Morrison formulas if we replace ω's and I's by the Dirac deltas.

4. HIGHER ORDER "METRIC" TENSORS

In this section we introduce several technical results which will turn our abstract composite index language into a practical tool.

We have seen that $\mathrm{Tr}(\hat{\rho}^2) = g^{ab}\rho_a\rho_b$. It is useful to introduce higher order tensors satisfying $\mathrm{Tr}(\hat{\rho}^n) = g^{a_1 \ldots a_n}\,\rho_{a_1} \ldots \rho_{a_n}$. Define

$$g^{a_1 \ldots a_n} = \omega^{\alpha_1\alpha'_n}\omega^{\alpha_2\alpha'_1}\omega^{\alpha_3\alpha'_2} \ldots \omega^{\alpha_{n-1}\alpha'_{n-2}}\omega^{\alpha_n\alpha'_{n-1}} \tag{22}$$

$$G_{a_1 \ldots a_n} = I_{\alpha_1\alpha'_n} I_{\alpha_2\alpha'_1} I_{\alpha_3\alpha'_2} \ldots I_{\alpha_{n-1}\alpha'_{n-2}} I_{\alpha_n\alpha'_{n-1}} \tag{23}$$

If we lower the indices in (22), we see that, somewhat counterintuitively, g does not go directly into G (although $g_{ab}\omega^b = I_a$!), but

$$g_{a_1b_1} \ldots g_{a_nb_n} g^{b_1 \ldots b_n} = g_{a_1 \ldots a_n} = G_{a_n a_{n-1} \ldots a_1} \tag{24}$$

So define a $*$-operation which reverses the order of indices: $** = \mathrm{id}$, $g^{*}_{a} = g_a$, $g^{*}_{ab} = g_{ba} = g_{ab}$, and

$$g^{*}_{a_1 \ldots a_n} = G_{a_1 \ldots a_n} = g_{a_n a_{n-1} \ldots a_1} \tag{25}$$

and similarly with the upper indices. The following properties are essential for further calculations.

(a) Cyclicity

$$g_{a_1\ldots a_n b_1\ldots b_m} = g_{b_1\ldots b_m a_1\ldots a_n} \tag{26}$$

$$g^*_{a_1\ldots a_n b_1\ldots b_m} = g^*_{b_1\ldots b_m a_1\ldots a_n} \tag{27}$$

(b) "Cut-and-glue"

$$g^*_{a_1\ldots a_n x} g^{*x}_{a_{n+1}\ldots a_{n+m}} = {}^*g_{a_1\ldots a_{n+m}} \tag{28}$$

$$g^{a_1\ldots a_n x} g^{a_{n+1}\ldots a_{n+m}}_{x} = g^{a_1\ldots a_{n+m}} \tag{29}$$

(c) "Annihilation"

$$I_x g^{a_1\ldots a_k x a_{k+1}\ldots a_{n+m}} = g^{a_1\ldots a_k a_{k+1}\ldots a_{n+m}} \tag{30}$$

$$\omega^x g_{a_1\ldots a_k x a_{k+1}\ldots a_{n+m}} = g_{a_1\ldots a_k a_{k+1}\ldots a_{n+m}} \tag{31}$$

(d) "Drag-and-drop"

$$g^{b_1\ldots b_1}_{x} g^{a_1\ldots a_k x a_{k+1}\ldots a_{n+m}} = g^{a_1\ldots a_k b_1\ldots b_l a_{k+1}\ldots a_{n+m}} \tag{32}$$

$$g^{x}_{b_1\ldots b_l} g_{a_1\ldots a_k x a_{k+1}\ldots a_{n+m}} = g_{a_1\ldots a_k b_1\ldots b_l a_{k+1}\ldots a_{n+m}} \tag{33}$$

It is practical to accept the rule stating that complex conjugation interchanges primed and unprimed indices. Assuming this, we can define symmetric operators $\hat{A}$ as those whose kernels satisfy $\overline{A_{\alpha\beta'}} = A_{\beta\alpha'}$. We find also that

$$\overline{g_{a_1\ldots a_n}} = g^*_{a_1\ldots a_n} \tag{34}$$

As a consequence,

$$\overline{g_{a_1\ldots a_n} A^{a_1}_{1} \ldots A^{a_n}_{n}} = g_{a_1\ldots a_n} A^{a_1}_{n} \ldots A^{a_n}_{1} \tag{35}$$

which is an abstract-index version of the well-known rule

$$\overline{\mathrm{Tr}(\hat{A}_1 \ldots \hat{A}_n)} = \mathrm{Tr}(\hat{A}_n \ldots \hat{A}_1) \tag{36}$$

valid for symmetric operators. In order to translate the abstract-index formulas into more standard operator ones, one uses the following correspondence:

$$(\hat{A}_1 \ldots \hat{A}_n)_a = g_{a a_1\ldots a_n} A^{a_1}_{1} \ldots A^{a_n}_{n} \tag{37}$$

5. STRUCTURE CONSTANTS REVISITED

A Lie–Nambu 3-bracket written in the form (4) is based on a totally antisymmetric 3-index tensor. Obviously, the tensor has 3-indices for all Lie algebras and for this reason it is not immediately clear whether a generalization of (4) to a "generalized Nambu" n-bracket is possible. On the other hand,

the structure constants occurring in (12) have a rich structure and it turns out there exists a natural generalization of (12).

To begin with, let us note that

$$\Omega_{abc} = g_{abc} - g_{acb} = 2!g_{a[bc]} = 2!g_{[abc]} \tag{38}$$

$$\Omega^{abc} = g^{abc} - g^{acb} = 2!g^{a[bc]} = 2!g^{[abc]} \tag{39}$$

where $[\ldots]$ denotes an antisymmetrization.

Consider

$$\Omega_{a_1\ldots a_n} = (n-1)!g_{[a_1\ldots a_n]} \tag{40}$$

Lemma 1. We have

$$g_{[a_1\ldots a_{2m}]} = 0 \tag{41}$$

$$g_{[xa_1\ldots a_{2m}]} = g_{x[a_1\ldots a_{2m}]} \tag{42}$$

$$\omega^x \Omega_{a_1\ldots x\ldots a_n} = 0 \tag{43}$$

Proof (a) Equation (41):

$$g_{[a_1\ldots a_n]} = g_{[a_2\ldots a_n a_1]} = (-1)^{n-1} g_{[a_1\ldots a_n]}$$

where the cyclicity and total antisymmetry were used. The expression vanishes for even n.

(b) Equation (42): Assume $n = 2m$. We have

$$\begin{aligned}
&g_{[xa_1\ldots a_n]} \\
&= \frac{1}{n+1}\left(g_{x[a_1\ldots a_n]} + \ldots + (-1)^k g_{[a_1\ldots a_k|x|a_{k+1}\ldots a_n]} + \ldots + (-1)^n g_{[a_1\ldots a_n]x}\right) \\
&= \frac{1}{n+1}\left(g_{x[a_1\ldots a_n]} + \ldots + (-1)^k g_{x[a_{k+1}\ldots a_n a_1\ldots a_k]} + \ldots + g_{x[a_1\ldots a_n]}\right) \\
&= \frac{1}{n+1}\left(g_{x[a_1\ldots a_n]} + \ldots + (-1)^{k+(n-k)k} g_{x[a_1\ldots a_n]} + \ldots + g_{x[a_1\ldots a_n]}\right) \\
&= g_{x[a_1\ldots a_n]}
\end{aligned}$$

where we have used the cyclicity and the fact that $(n-k+1)k$ is even for any k if n is even.

(c) Equation (43): It is sufficient to note that the annihilation property together with (41) and (42) imply

$$\omega^x g_{[xa_1 \ldots a_{2m}]} = \omega^x g_{x[a_1 \ldots a_{2m}]} = g_{[a_1 \ldots a_{2m}]} = 0 \quad \blacksquare$$

6. GENERALIZED LIE–NAMBU BRACKETS

We define the generalized Lie–Nambu bracket for $n = 2m + 1$ by

$$\{A_1, \ldots, A_n\} = \Omega_{a_1 \ldots a_n} \frac{\delta A_1}{\delta \rho_{a_1}} \cdots \frac{\delta A_n}{\delta \rho_{a_n}} \tag{44}$$

Let $C_k = g^{a_1 \ldots a_k} \rho_{a_1} \ldots \rho_{a_k}$.

Theorem 1. We have

$$\{C_{k_1}, \ldots, C_{k_{(n+1)/2}}, \cdot, \ldots, \cdot\} = 0 \tag{45}$$

Proof. Let us begin with the following remark. Assume a tensor $F_{\ldots abc\ldots}$ has the "drag-and-drop" property

$$F_{\ldots abc\ldots} g^{\ c}_{c_1 c_2} = F_{\ldots abc_1 c_2 \ldots}$$

and consider

$$\begin{aligned} & F_{\ldots [abc] \ldots} \rho^a g^{\ c}_{c_1 c_2} \rho^{c_1} \rho^{c_2} \\ & = \frac{1}{6} (F_{\ldots abc\ldots} + F_{\ldots bca\ldots} + F_{\ldots cab\ldots} - F_{\ldots acb\ldots} - F_{\ldots cba\ldots} \\ & \quad - F_{\ldots bac\ldots}) \rho^a g^{\ c}_{c_1 c_2} \rho^{c_1} \rho^{c_2} \\ & = \frac{1}{6} (F_{\ldots abc_1 c_2 \ldots} + F_{\ldots bc_1 c_2 a\ldots} + F_{\ldots c_1 c_2 ab\ldots} - F_{\ldots ac_1 c_2 b\ldots} \\ & \quad - F_{\ldots c_1 c_2 ba\ldots} - F_{\ldots bac_1 c_2 \ldots}) \rho^a \rho^{c_1} \rho^{c_2} \\ & = \frac{1}{6} (F_{\ldots abc_1 c_2 \ldots} - F_{\ldots c_1 c_2 ba\ldots}) \rho^a \rho^{c_1} \rho^{c_2} \end{aligned} \tag{46}$$

which in general does not vanish. We can see therefore that for expressions such as (46) to vanish, it is sufficient to have in each term of the decomposition (46) at least two transvected indices which are not seperated by a non-transvected one. With this observation in mind consider

$$g_{x[a_1 \ldots a_N c_1 \ldots c_M]} g^{\ a_1}_{a_1^1 \ldots a_1^{K_1}} \ldots g^{\ a_N}_{a_N^1 \ldots a_N^{K_N}} \rho^{a_1^1} \ldots \rho^{a_1^{K_1}} \ldots \rho^{a_N^1} \ldots \rho^{a_N^{K_1}} \tag{47}$$

Expanding (47), we obtain a sum involving expressions $g_{x\ldots ij\ldots}$ and $g_{x\ldots ji\ldots}$

entering with opposite signs. If N and M are chosen in a way to guarantee that for any such term there exists at least a pair (a_k, a_l) of indices that are not separated by some c_r, then (47) vanishes on the basis of the preceding argument. We know that $N + M$ is an even number. Therefore the greatest N that allows for a separation of any two a_k is $N = (N + M)/2 = M = (n - 1)/2$. To complete the proof, it is sufficient to note that (47) equals

$$\frac{1}{k_1} \cdots \frac{1}{k_n} \{C_{k_1}, \ldots, C_{k_N}, \cdot, \ldots, \cdot\} \tag{48}$$

where $k_j = K_j + 1$. ■

Remarks. (1) Total antisymmetry of (44) guarantees that $\{C_k, C_k, \ldots\} = 0$. Moreover, (43) leads to $\{C_1, \ldots\} = 0$. Since $C_1 = \mathrm{Tr}\,\hat{\rho}$, the dynamics generated by such brackets is trace preserving.

(2) For any k, l, we have $\{C_k, C_l, \ldots\}|_{\text{purestates}} = 0$. This follows from

$$g^{ab_1 \ldots b_n} \rho_{b_1} \ldots \rho_{b_n}\big|_{\text{purestates}} = \rho^a C_1^{n-1}\big|_{\text{purestates}} \tag{49}$$

(3) Theorem 1 was proved for $n = 3$ in Czachor (1997a).

Theorem 2. Let $S_j = S_j(C_1, C_2, \ldots)$ be a differentiable function of C_k, $k = 1, 2, 3, \ldots$, and z_n a complex number. The dynamics given by

$$\dot{\rho}_a = z_n \{\rho_a, H_1, \ldots, H_{(n-1)/2}, S_1, \ldots, S_{(n-1)/2}\} \tag{50}$$

conserves C_k. The C_k are Casimir invariants, i.e.,

$$\{C_k, A_1, \ldots, A_{(n-1)/2}, S_1, \ldots, S_{(n-1)/2}\} = 0 \tag{51}$$

for any functions A_k.

Remarks. (1) Theorem 2 is a straightforward consequence of Theorem 1.

(2) The number z_n will be assumed to satisfy $\bar{z}_n = -z_n$ (for $n = 4m + 3$) or $\bar{z}_n = z_n$ (for $n = 4m + 1$), $m = 0, 1, 2, \ldots$. The simplest choice is therefore either $z_n = -i$, for $n = 4m + 3$, or $z_n = 1$, for $n = 4m + 1$ (see the discussion below).

7. 3-BRACKET

The simplest $n = 4m + 3$ case is $n = 3$. The discussion given by Białynicki-Birula and Morrison dealt with *linear* quantum mechanics. The possibility of using the 3-bracket dynamics as a departure point for *nonlinear* generalizations of quantum mechanics was described in some detail in Czachor (1997a). One of the main motivations for studying the 3-bracket dynamics was the possibility of introducing nonlinearities only by generalizations

of S and without modifications of H. Generalizations via nonlinear H are interesting and will be discussed in the next sections. An important drawback of such Hamiltonian generalizations is that we have to represent observables by nonlinear operators, which leads to interpretational difficulties. To give an example, it is not clear which definition of a nonlinear eigenvalue is physically meaningful, or how to represent higher moments of experimentally measured random variables if nonlinear operators are involved (Czachor, 1996a). Let us therefore first consider what happens if $H = H^a \rho_a = \mathrm{Tr}\, \hat{H}\hat{\rho}$ and S is an arbitrary (differentiable) function of the Casimirs C_k. It is easy to see that the dynamics given by a 3-bracket is then linear if and only if S is linear in C_2 (Czachor, 1997a).

7.1. Lie–Nambu Duality

Does the 3-bracket lead to a Poisson bracket? The answer to this question reveals an interesting duality which points to two different generalizations of linear quantum mechanics. To understand the problem define $\{A, B\}_X := \{A, B, X\}$ and check whether the Jacobi identity is satisfied. Consider

$$\{\{A, B\}_X, C\}_X + \{\{C, A\}_X, B\}_X + \{\{B, C\}_X, A\}_X$$
$$= \frac{\delta A}{\delta\rho_d}\frac{\delta B}{\delta\rho_e}\frac{\delta^2 X}{\delta\rho_a\delta\rho_f}\frac{\delta C}{\delta\rho_b}\frac{\delta X}{\delta\rho_c}(\Omega_{def}\Omega_{abc} + \Omega_{bdf}\Omega_{aec} + \Omega_{ebf}\Omega_{adc}) \quad (52)$$

The terms involving second derivatives of A, B, and C drop out just because of the total antisymmetry of structure constants. The term involving the second drivative of X vanishes in several cases. For $X = S = g^{ab}\rho_a\rho_b/2$ the second derivative gives g^{af} and (52) vanishes on the basis of the structure constants version of the Jacobi identity. With this choice of X the bracket $\{\cdot,\cdot\}_S$ is a Lie–Poisson bracket and the dynamics given by

$$\dot{\rho}_a = -i\{\rho_a, H\}_S = -i\{\rho_a, H, S\} \quad (53)$$

is an ordinary Lie–Poisson dynamics. If H is nonlinear, the dynamics corresponds to the nonlinear quantum mechanics in the Bóna–Jordan version (Bóna, 1991; Jordan, 1993). It can be shown that such brackets satisfy the Jacobi identity for all $S = S(C_2)$ (Czachor, 1996b). If S is a function of higher order Casimirs, say, $S = C_3$, the Jacobi identity does not hold. However, rewriting (53) as

$$\dot{\rho}_a = -i\{\rho_a, S\}_{-H} = -i\{\rho_a, S, -H\} \quad (54)$$

we obtain a Poisson bracket for any S if $X = -H$ is *linear*. It follows that the requirement that *observables* are linear leads us, via the Lie–Nambu 3-bracket, to a dual Poisson structure given by (54). This *Lie–Nambu duality*

$(H, S) \leftrightarrow (S, -H)$ is typical of all Lie–Nambu theories and is analogous to the canonical $(q, p) \leftrightarrow (p, -q)$ and electromagnetic $(E, B) \leftrightarrow (B, -E)$ dualities. The duality transformation is a particular case of the duality rotation

$$\{\cdot, H, S\} = \{\cdot, H\cos\alpha - S\sin\alpha, H\sin\alpha + S\cos\alpha\} \tag{55}$$

It is noteworthy that since the X-bracket does not in general satisfy the Jacobi identity, the 3-bracket cannot satisfy the so-called fundamental identity discussed in Takhtajan (1994). The example of the Lie–Nambu duality, where the Jacobi identity simultaneously holds and does not hold (depending on the viewpoint), clearly shows that status of such identities is more technical than fundamental. Each kind of dynamics seems to have its own fundamental criteria of sensibility. In this work we insist on the positivity of density matrices and lack of faster-than-light effects.

Of interest in the context of the duality is the pure-state case where $C_n = \langle\psi|\psi\rangle^n$, $S = S(\|\psi\|)$, and $H = \langle\psi|\hat{H}|\psi\rangle$. The Hamilton equations (14) can be written as

$$i\dot{\psi}_\alpha = H_{\alpha\alpha'}\frac{\delta I}{\delta\overline{\psi}_{\alpha'}}, \qquad -i\dot{\overline{\psi}}_{\alpha'} = H_{\alpha\alpha'}\frac{\delta I}{\delta\psi_\alpha} \tag{56}$$

where $I = \omega^{\alpha\alpha'}\psi_\alpha\overline{\psi}_{\alpha'} = \|\psi\|^2$. If $\psi = \psi_{AB}$ is the electromagnetic spinor, its squared norm $\|\psi\|^2$ is equal to the classical energy of the field. Therefore here H is *not* the energy, although formally it is an expression analogous to the Dirac Hamiltonian function (Białynicki-Birula, 1996). It is known that the Hamiltonian formalism based on the energy density $|\psi(x)|^2 = \boldsymbol{E}(x)^2 + \boldsymbol{B}(x)^2$ corresponds to the Poisson tensor which involves the differential operator $\vec{J}\cdot i\vec{\nabla} = \vec{\nabla}\times$ (Białynicki-Birula and Białynicka-Birula, 1976). On the other hand, taking $H = \langle\psi|\vec{J}\cdot i\vec{\nabla}|\psi\rangle$ as the Hamiltonian function, one gets the Poisson tensor which involves no differentiations. The nonlinear electrodynamics of the Born–Infeld type (Plebański, 1970) may be regarded as a nonlinear generalization within the $\{\cdot, S\}_{-H}$ scheme.

Another manifestation of the duality can be seen in a simpler case of N harmonic oscillators. The classical energy $E = \Sigma\,(p_k^2 + q_k^2) = \Sigma\,|p_k + iq_k|^2 = \Sigma\,|\psi_k|^2$ can be regarded as a *norm* squared in the Hilbert space $\boldsymbol{C}^N$. It is an easy exercise to rewrite the equations of motion as an N-dimensional Schrödinger equation with $\hat{H}$ being a diagonal matrix whose eigenvalues are the energies of the oscillators, but then $H = \langle\psi|\hat{H}|\psi\rangle$ is not the energy.

The above facts suggest an alternative interpretation of the Poisson structures that occur in quantum mechanics: A modification of H (say, by interactions) can be understood as a deformation of the Poissonian structure $\{\cdot\,,\cdot\}_{-H}$ of the manifold of states, and not as a modification of a Hamiltonian function. Keeping H unchanged, but modifying S, one changes a flow on the Poisson manifold, but the structure of the manifold itself is unchanged.

7.2. Canonical Transformations

Nonlinear quantum mechanics based on $\{\cdot\,,\,\cdot\}_S$ uses nonlinear H and $S = S(C_2)$. The canonical transformations must therefore be those that do not change $C_2 = \mathrm{Tr}(\hat{\rho}^2)$ (or $\langle\psi|\psi\rangle$ for pure states). Such transformations can be nonlinear and were discussed by Weinberg (1989) and Jordan (1993). The version based on $\{\cdot\,,\,\cdot\}_{-H}$ leads to canonical transformations that keep $H = \mathrm{Tr}\hat{H}\hat{\rho}$ linear (or $\langle\psi|\hat{H}|\psi\rangle$ bilinear for pure states). The two classes of transformations are not equivalent. It is natural to require that only those observables which commute with H have to be represented by linear functionals. Various Poissonian structures that appear in this context may be used also for a combined quantum–classical description, as shown by Jones (1993, 1994) for the Weinberg-type theory.

7.3. Formal Solutions

Consider first the dynamics with $S = C_{k+1}/(k+1)$. The 3-bracket equation

$$\dot{\rho}_a = -i\{\rho_a, H, S\} = -i\Omega_{abc}H^b g^{cc_1\ldots c_k}\rho_{c_1}\ldots\rho_{c_k} \tag{57}$$

has solutions which can be formally written as

$$\rho_a(t) = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\underbrace{\{\ldots\{\{\rho_a(0), H, S\}, H, S\}\ldots, H, S\}}_{n} \tag{58}$$

Let $\rho_a(0) = \overline{\rho_a(0)}$. In the standard notation we have $\hat{\rho}(0) = \hat{\rho}(0)^*$ and

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}^k] \tag{59}$$

For $\hat{\rho} = \hat{\rho}^2$, (59) is the ordinary linear Liouville–von Neumann equation. Asuming $\hat{H}^* = \hat{H}$, we find $\hat{\rho}(0)^* = \hat{\rho}(0)$. In the same way we can prove that $d^n\hat{\rho}/dt^n|_{t=0} = (d^n\hat{\rho}/dt^n|_{t=0})^*$. It follows that the formal solution satisfies $\hat{\rho}(t) = \hat{\rho}(t)^*$ if $\hat{\rho}(0) = \hat{\rho}(0)^*$. The same argument applies to more general $S = S(C_1, C_2, \ldots)$. Equation (59) is interesting in itself even in finite-dimensional cases, where the above argument can be made more rigorous. To show that the spectrum of self-adjoint Hilbert–Schmidt solutions of the 3-bracket equations of motion is conserved by the 3-bracket dynamics, one uses the following result.

Lemma 2. Consider a sequence of probabilities $\{p_k\}_{k=0}^{\infty}$ and an arbitrary real sequence $\{a_k\}_{k=0}^{\infty}$ satisfying for any natural n

$$\sum_{k=0}^{\infty}p_k^n = \sum_{k=0}^{\infty}a_k^n \tag{60}$$

Then the two sequences are identical up to permutation.

Remarks. (1) Let the two sequences represent spectra of a Hermitian Hilbert–Schmidt solution of the n-bracket equation at $t = 0$ and $t \neq 0$, respectively. Since $t \mapsto \hat{\rho}(t)$ is continuous, the spectrum of $\hat{\rho}(t)$ is also continuous and, hence, conserved. The condition (60) is implied by conservation of C_n.

(2) Lemma 2 was proved in Czachor and Marciniak (1997).

(3) Such solutions can be interpreted as nonlinearly evolving density matrices. The question of their *complete* positivity will be discussed below.

8. 5-BRACKET

The case $n = 5$ is the simplest (nontrivial) $4m + 1$ case. The equation of motion is ($z_5 = 1$)

$$\dot{\rho}_a = \{\rho_a, H_1, H_2, S_1, S_2\} \tag{61}$$

Assume that H_k are linear in ρ. The simplest choice of the other two generators is $S_1 = C_2/2$, $S_2 = C_3/3$. Nontrivial 5-bracket equations of motion (61) are always nonlinear, as opposed to the 3-bracket ones, which can be linear, and always vanish on pure states. The RHS of (61) when written in the standard notation involves an antisymmetrized product of $\hat{H}_1$, $\hat{H}_2$, $\hat{\rho}$, and $\hat{\rho}^2$. After a simplification one finds

$$\begin{aligned}\dot{\hat{\rho}} = {} & ([\hat{\rho}, \hat{H}_1]\hat{H}_2 - [\hat{\rho}, \hat{H}_2]\hat{H}_1)\hat{\rho}^2 + \hat{\rho}^2(\hat{H}_2[\hat{H}_1, \hat{\rho}] - \hat{H}_1[\hat{H}_2, \hat{\rho}]) \\ & + \hat{\rho}(\hat{H}_2\hat{\rho}^2\hat{H}_1 - \hat{H}_1\hat{\rho}^2\hat{H}_2) + (\hat{H}_1\hat{\rho}^2\hat{H}_2 - \hat{H}_2\hat{\rho}^2\hat{H}_1)\hat{\rho}\end{aligned} \tag{62}$$

The RHS of (62) is Hermitian if $\hat{\rho}$, $\hat{H}_1$, and $\hat{H}_2$ are Hermitian. This explains the choice of *real* z_5. For $\hat{\rho} = \hat{\rho}^2$, (62) vanishes. Assuming that $\rho(0) = \rho(0)^*$, we find that all higher derivatives are also Hermitian. The formal solution

$$\rho_a(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\{\ldots\{\{\rho_a(0), H_1, H_2, S_1, S_2\}, H_1, H_2, S_1, S_2\} \ldots, H_1, H_2, S_1, S_2\}}_{n} \tag{63}$$

satisfies $\rho(t) = \rho(t)^*$ if $\rho(0) = \rho(0)^*$. Using the same argument as for $n = 3$, we conclude that the spectrum of self-adjoint and Hilbert–Schmidt solutions of (61) is conserved.

9. *N*-PARTICLE EXTENSIONS OF ONE-PARTICLE ALMOST-LIE–POISSON DYNAMICS

An extension of dynamics from 1 to N particles is a delicate problem. Careful analysis shows that the Lie–Nambu duality holding for the 3-brackets

leads to generalizations which behave differently from the viewpoint of N-particle extensions. The Poisson dynamics based on $\{\cdot\,,\,\cdot\}_s$ for $S = C_2/2$ is the most regular one. An inclusion of nonlinear Hamiltonian functions H does not lead to difficulties with independent evolutions of separated systems. This fact was proved by Polchinski (1991) and Jordan (1993) and recently generalized by myself to those nonlinear theories which do not possess Hamiltonian *functions*, but only Hamiltonian *operators* (Czachor, 1997b). In spite of this, the view that *any* nonlinear generalization of a Schrödinger dynamics leads to problems with causality is quite popular. Nonlinear quantum mechanics based on $\{\cdot\,,\,\cdot\}_{-H}$ leads to a new kind of nonlocal phenomenon. This effect, typical of mixed states, is analogous to the threshold phenomena discussed by Goldin and Svetlichny (1994) for pure states.

9.1. *N*-Particle "Metric" Tensors

Let $g_{a_1 \ldots a_n}$ be a one-particle metric tensor. The N-particle tensor is defined by

$$g^{N}_{a_1 \ldots a_n} = g_{a_1^1 \ldots a_n^1} \ldots g_{a_1^N \ldots a_n^N} \tag{64}$$

The indices on the LHS of (64) are the N-particle ones: $a_1 = a_1^1 \ldots a_1^N$, etc. The N-particle n-bracket is defined by

$$\{A_1, \ldots, A_n\}^N = \Omega^{N}_{a_1 \ldots a_n} \frac{\delta A_1}{\delta \rho^N_{a_1}} \ldots \frac{\delta A_n}{\delta \rho^N_{a_n}} \tag{65}$$

where $\rho^N_a = \rho_{a^1 \ldots a^N}$,

$$\Omega^{N}_{a_1 \ldots a_n} = (n-1)!\; g^{N}_{[a_1 \ldots a_n]} \tag{66}$$

For identical particles (bosons and fermions), $\rho_{a^1 \ldots a^N} = \rho_{(a^1 \ldots a^N)}$ with $(\ldots)$ denoting symmetrization. A distinction between fermions and bosons can be seen at the "spinor index" level:

$$\rho_{a^1 \ldots a^N} = \rho_{[\alpha^1|\alpha'^1|\ldots|\alpha^N]\alpha'^N} = \rho_{\alpha^1[\alpha'^1|\ldots|\alpha^N|\alpha'^N]} \qquad \text{(fermions)} \tag{67}$$

$$\rho_{a^1 \ldots a^N} = \rho_{(\alpha^1|\alpha'^1|\ldots|\alpha^N)\alpha'^N} = \rho_{\alpha_1(\alpha'^1|\ldots|\alpha^N|\alpha'^N)} \qquad \text{(bosons)} \tag{68}$$

For the 3-bracket the annihilation property implies the important identity

$$\Omega^{N}_{abc}\omega^{b_1} \ldots \omega^{b_{k-1}}\,\omega^{b_{k+1}} \ldots \omega^{b_N} = g_{a_1 c_1} \ldots g_{a_{k-1} c_{k-1}} \Omega_{a_k b_k c_k}\, g_{a_{k+1}\, c_{k+1}} \ldots g_{a_N c_N} \tag{69}$$

where $\Omega_{a_k b_k c_k}$ are the one-particle structure constants of the kth particle.

9.2. Extension of Hamiltonians

The 3-bracket dynamics is Lie–Poisson if $S = C_2/2$ and a Hamiltonian function H exists. There exist nonlinear Liouville–von Neumann equations that possess Hamiltonian operators of the form $\hat{H}(\rho) = \hat{H}_1(\rho) + \hat{H}_2(\rho)$ where $\mathrm{Tr}\ \hat{H}_2(\rho)\hat{\rho} = 0$. Such equations do not possess a Hamiltonian function $\mathrm{Tr}\ \hat{H}(\rho)\hat{\rho}$, but often are of physical interest [e.g., starting with nonlinearizable Doebner–Goldin Schrödinger equations (Doebner and Goldin, 1996), one arrives at this class of mixed state equations (Czachor, 1997b)]. Equations that can be written as

$$\dot{\rho}_a = -i\Omega_{abc}H^b(\rho)\rho^c \tag{70}$$

although no H satisfying $H^b = \dfrac{\delta H}{\delta \rho_b}$ exists, will be called almost-Lie–Poisson. Assume we have N (not necessarily identical) particles that do not interact with one another (but can interact with something else and do not have to be free). Each of them satisfies a one-particle equation (70) with some H. We define the N-particle extension of (70) by

$$\rho^N_a = -i\Omega^N_{abc}H^b(\rho^N)\rho^{Nc}, \tag{71}$$

where

$$H^b(\rho^N) = H^{b_1}_1(\rho_{(1)})\omega^{b_2}\ldots\omega^{b_N} + \ldots + \omega^{b_1}\ldots\omega^{b_{N-1}}H^{b_N}_N(\rho_{(N)}) \tag{72}$$

The reduced density matrix $\rho_{(k)}$ is defined by

$$\rho_{(k)a_k} = \omega^{a_1}\ldots\omega^{a_{k-1}}\omega^{a_{k+1}}\ldots\omega^{a_N}\rho_{a_1\ldots a_k\ldots a_N} \tag{73}$$

There are two motivations for (72). First, if the kth particle is described by a Hamiltonian function $H_k(\rho) = H_k(\rho_{(k)})$, then (72) is just a consequence of the chain rule for functional derivatives. The second motivation is (69). Indeed, applying (69) to (71), we obtain

$$\begin{aligned} i\dot{\rho}_{a_1\ldots a_N} &= \Omega_{a_1b_1c_1}H^{b_1}_1(\rho_{(1)})\rho^{c_1}{}_{a_2\ldots a_N} + \ldots \\ &\quad + \Omega_{a_Nb_Nc_N}H^{b_N}_N(\rho_{(N)})\rho_{a_1\ldots a_{N-1}c^N} \end{aligned} \tag{74}$$

Transvecting both sides of (74) with

$$\omega^{a_1}\ldots\omega^{a_{k-1}}\omega^{a_{k+1}}\ldots\omega^{a_N}$$

and using $\omega^a\Omega_{abc} = 0$, we get

$$\dot{\rho}_{(k)a} = -i\Omega_{a_kb_kc_k}H^{b_k}_k(\rho_{(k)})\rho^{c_k}_{(k)} \tag{75}$$

Both sides of (75) depend only on objects which are intrinsic to the kth

subsystem. It follows that the reduced density matrix of this subsystem "does not see" the other noninteracting systems. *The observers in the other subsystems have no possibility of influencing the dynamics of the kth one by any kind of modification of the Hamiltonians in the other separated systems.* In particular, they cannot influence any observable quantity in the *k*th subsystem by different choices of measurements in their "own" subsystems. This explicitly contradicts the popular claim that any nonlinear dynamics must imply faster-than-light influences between separated systems.

Denoting the dynamics of the *N*-particle system by ϕ_N^t, the one corresponding to the *k*th subsystem by ϕ_k^t, and by Tr_{N-k} the partial trace which reduces the dynamics from the composite system to the *k*th one, we get an important separability condition

$$\mathrm{Tr}_{N-k} \circ \phi_N^t = \phi_k^t \circ \mathrm{Tr}_{N-k} \tag{76}$$

characteristic of the Lie–Poisson dynamics. The dynamics satisfying (76) and (75) can be termed *completely separable*.

9.3. Examples of Completely Separable Extensions

The method of extension given by (72) applies to any equation whose one-particle Hamiltonian operator can be written as a function of the particle's *density matrix*. This applies also to pure-state (Schrödinger) equations. To see how this works for nonlinear Schrödinger equations, consider some examples. A Hamiltonian operator consists of two parts: a linear part $\hat{H}_L(x) = \hat{H}_{\text{kinetic}} + V(x)$ and a nonlinear part $\hat{H}_{NL} = \hat{H}(\psi, \bar{\psi}; x)$. To apply the above method we have to be able to write $\hat{H}(\psi, \bar{\psi}; x)$ as $\hat{H}(\rho; x)$.

(a) *"Nonlinear Schrödinger equation"*:

$$\hat{H}(\psi, \bar{\psi}; x) = |\psi(x)|^2 \to \hat{H}(\rho; x) = \rho(x, x)$$

(b) *Białynicki-Birula–Mycielski equation* (Białynicki-Birula and Mycielski, 1976):

$$\hat{H}(\psi, \bar{\psi}; x) = \ln(|\psi(x)|^2) \to \hat{H}(\rho; x) = \ln \rho(x, x)$$

Obviously in the same way one can treat any equation with nonlinearities given by some function $H(|\psi(x)|)$.

(c) *Haag–Bannier equation* (Haag and Bannier, 1978):

$$\hat{H}(\psi, \bar{\psi}; x) = \overline{\overline{A}}(x) \frac{\bar{\psi}(x)\overline{\overline{\nabla}}_x \psi(x) - \psi(x)\overline{\overline{\nabla}}_x \bar{\psi}(x)}{2i|\psi(x)|^2} \to$$

$$\hat{H}(\rho; x) = \overline{\overline{A}}(x) \frac{\int dz\, \delta(x - z)\overline{\overline{\nabla}}_x [\rho(x, z) - \rho(z, x)]}{2i\rho(x, x)} \tag{77}$$

(d) *Doebner–Goldin equations* (Doebner and Goldin, 1996; Nattermann, 1997). There are five nonlinear terms denoted by R_k:

$$R_1(\psi, \bar{\psi}; x) = \frac{1}{2i} \frac{\bar{\psi}(x)\Delta_x\psi(x) - \psi(x)\Delta_x\bar{\psi}(x)}{|\psi(x)|^2} \rightarrow$$

$$R_1(\rho; x) = \frac{1}{2i} \frac{\int dz\, \delta(x - z)\Delta_x [\rho(x, z) - \rho(z, x)]}{\rho(x, x)}$$

$$R_2(\psi, \bar{\psi}; x) = \frac{\Delta_x|\psi(x)|^2}{|\psi(x)|^2} \rightarrow R_2(\rho; x) = \frac{\Delta_x\rho(x, x)}{\rho(x, x)}$$

$$R_3(\psi, \bar{\psi}; x) = \frac{1}{(2i)^2} \frac{[\bar{\psi}(x)\overline{\nabla}_x\psi(x) - \psi(x)\overline{\nabla}_x\bar{\psi}(x)]^2}{|\psi(x)|^4} \rightarrow$$

$$R_3(\rho; x) = \frac{1}{(2i)^2} \frac{\left(\int dz\, \delta(x - z)\overline{\nabla}_x [\rho(x, z) - \rho(z, x)]\right)^2}{\rho(x, x]^2}$$

$$R_4(\psi, \bar{\psi}; x) = \frac{1}{2i} \frac{[\bar{\psi}(x)\overline{\nabla}_x\psi(x) - \psi(x)\overline{\nabla}_x\bar{\psi}(x)] \cdot \overline{\nabla}_x|\psi(x)|^2}{|\psi(x)|^4}$$

$$R_4(\rho; x) = \frac{1}{2i} \frac{\int dz\, \delta(x - z)\overline{\nabla}_x [\rho(x, z) - \rho(z, x)] \cdot \overline{\nabla}_x\rho(x, x)}{\rho(x, x)^2}$$

$$R_5(\psi, \bar{\psi}; x) = \frac{[\overline{\nabla}_x|\psi(x)|^2]^2}{|\psi(x)|^4} \rightarrow R_5(\rho; x) = \frac{[\overline{\nabla}_x\rho(x, x)]^2}{\rho(x, x)^2}$$

(e) *Twarock equation* on S^1 (Twarock, 1997):

$$\hat{H}(\psi, \bar{\psi}; x) = \frac{\psi(x)''\overline{\psi(x)'} - \overline{\psi(x)''}\psi(x)'}{\psi(x)\overline{\psi(x)'} - \overline{\psi(x)}\,\psi(x)'} \rightarrow$$

$$\hat{H}(\rho; x) = \frac{\left[\int dy\, \delta(x - y)\partial_x^2\rho(x, y)\right]\left[\left[\int dz\, \delta(x - z)\partial_x^2\rho(z, x)\right] - c.c.\right]}{\rho(x, x)\int dy\, \delta(x - y)\partial_x\rho(x, y) - c.c.}$$

(f) (n, n)-*homogeneous nonlinearities*. Denote by D a differential operator involving arbitrary mixed partial derivatives up to order k. Consider a real function $H(\psi) = F(D\psi(x))$, which is (n, n)-homogeneous, i.e., satisfies $H(\lambda\psi) = \lambda^n\bar{\lambda}^n H(\psi)$. We first write

$$F(D\psi(x)) = \frac{F(\overline{\psi(x)}D\psi(x))}{|\psi(x)|^{2n}}$$

and then apply the tricks used for the Haag–Bannier, Doebner–Goldin, and Twarock terms. Obviously any reasonable function of such (n, n)-homogeneous expressions with different n's will do as well.

Let us now concentrate on the simplest case with $H_k(\rho; x) = H_k(\rho(x, x))$ and just two particles. The two-particle extension of the nonlinear part of the Hamiltonian is

$$\hat{H}_1(\rho_{(1)}(x_1, x_1)) + \hat{H}_2(\rho_{(2)}(x_2, x_2)) = H_1\left(\int dy\ \rho(x_1, y, x_1, y)\right) + H_2\left(\int dy\ \rho(y, x_2, y, x_2)\right) \quad (78)$$

If the two-particle state is pure, $\rho(x_1, x_2, x_1', x_2') = \Psi(x_1, x_2)\,\Psi^*(x_1', x_2')$, the RHS becomes

$$H_1\left(\int dy\ |\Psi(x_1, y)|^2\right) + H_2\left(\int dy\ |\Psi(y, x_2)|^2\right) \quad (79)$$

and reduces to

$$H_1(|\psi(x_1)|^2) + H_2(|\phi(x_2)|^2) \quad (80)$$

on product states $\Psi(x_1, x_2) = \psi(x_1)\phi(x_2)$.

It seems that an example that cannot be treated in this way is the Kostin equation (Kostin, 1972) involving the nonlinearity $\ln[\psi(x)/\psi(x)]$.

The reader may have noticed that the above reasoning involves two "heretical" elements. First of all the two-particle extension of dynamics for *nonfactorizable* (entangled) states leads to integrodifferential equations. Such equations are typically rejected in the nonlinear quantum mechanics literature as *nonlocal*. The construction presented above shows that the situation is in fact just the opposite. The requirement of locality (complete separability) leads us to *appropriate* integral terms and precisely because of these terms the subsystems can be completely isolated from one another. Second, *all* nonlinearities of the form $F(|\psi(x)|^2)$ are acceptable. This is in an apparent contradiction with the well-known result of Białynicki-Birula and Mycielski, who used the separability criterion to *derive* the logarithmic nonlinearity. However, they assumed that the two-particle extension has to be $F(|\Psi(x_1, x_2)|^2)$ and, with the condition

$$F(|\psi(x_1)\phi(x_2)|^2) = F(|\psi(x_1)|^2) + F(|\phi(x_2)|^2) \quad (81)$$

they found that only $F \sim \ln$ is acceptable. One obvious drawback of such

extensions is that they do not tell us what to do if the systems are noninteracting, but correlated and $\Psi(x_1, x_2)$ does not factorize. In such a case *local* probability densities are obtained by integrating out the coordinates of the remaining particles and it is quite logical that such expressions occur in the N-particle "correctly extended" Hamiltonians we discussed.

9.4. Problem of Complete Positivity

A subsystem described by $\rho_{(k)}$ can be embedded into a composite one described by ρ^N in a way guaranteeing the consistency of (71) and (75). The dynamics of $\rho_{(k)}$ is independent of N. In addition, since both $\rho_{(k)}$ and ρ^N satisfy the 3-bracket Lie–Poisson equation, the extension procedure preserves the positivity of dynamics at both subsystem and composite system levels. A dynamics that has these properties is typically associated with the notion of a *completely* positive map, provided the maps are *linear*.

In the mathematical literature the notion of complete positivity is generalized to nonlinear maps in a way that can be translated to our context as follows (Ando and Choi, 1986; Arveson, 1987; Majewski, 1990; Alicki and Majewski, 1990). One takes a positive map

$$\phi_1^t(a) = a(t), \qquad \phi_1^t: \quad \mathcal{A} \to \mathcal{A} \tag{82}$$

where $\mathcal{A}$ is a unital C^*-algebra. In our case $a = \rho_{(k)}$ and $\phi_1^t(a) = \rho_{(k)}(t)$. Assume for simplicity that the dimension of the kth system is finite. In the next step one considers a density matrix $\rho^N(0)$ of a bigger system consisting of the original one plus a system which has a finite number m of degrees of freedom. Writing

$$\rho^N(0) = \sum_{r,r',s,s'} \rho^N(0)_{rr'ss'} |r\rangle\langle r'| \otimes |s\rangle\langle s'| = \sum_{s,s'} a_{ss'} \otimes |s\rangle\langle s'| \tag{83}$$

we can represent $\rho^N(0)$ by the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \tag{84}$$

whose entries are elements of $\mathcal{A}$. The (nonlinear) map ϕ_1^t is said to be completely positive if the matrix

$$\begin{pmatrix} \phi_1^t(a_{11}) & \cdots & \phi_1^t(a_{1m}) \\ \vdots & \vdots & \vdots \\ \phi_1^t(a_{m1}) & \cdots & \phi_1^t(a_{mm}) \end{pmatrix} \tag{85}$$

is positive for any m. This is equivalent to the positivity of

$$\tilde{\phi}^t(\rho^N(0)) = \sum_{s,s'} \phi_1^t(a_{ss'}) \otimes |s\rangle\langle s'| \tag{86}$$

However, for nonlinear ϕ_1^t the explicit form of (86) for $t > 0$ is different for differenet choices of bases $\{|s\rangle\}$, which is unphysical unless there exists a superselection rule distinguishing a particular basis. In the generic case no such distinguished basis exists. Therefore a *basis-independent* extension from one to more particles cannot have the forms (85) and (86). And, indeed, the dynamics following from the Lie–Poisson extension discussed above does not coincide with (86). This was shown by an explicit calculation in Czachor and Kuna (1997b) but could be inferred also from the basis independence of the N-particle extension. It must be stressed that the dynamics (86) is the one that was used by Gisin (1989) in his discussion of unphysical influences between separated systems.

10. NONLOCAL PROPERTIES OF N-PARTICLE EXTENSIONS FOR THE DUAL POISSON STRUCTURE $\{\cdot\,,\cdot\}_{-H}$

The regularity of the N-particle extensions typical of an almost-Lie–Poisson dynamics is lost when one considers the dual Poisson structure $\{\cdot\,, S\}_{-H}$ with S a higher order Casimir invariant. To explicitly see the kind of difficulties one may encounter, consider the two-particle equation

$$i\dot{\rho}_{a_1a_2} = \{\rho_{a_1a_2}, C_3/3\}_{-H} \tag{87}$$

General properties of the 3-bracket dynamics imply that $C_n(\hat{\rho})$ are conserved for any natural n, where $\hat{\rho}$ is the two-particle density matrix. Also $C_1(\hat{\rho}_{(1)})$ is a constant of motion. However,

$$i\dot{C}_2(\hat{\rho}_{(1)}) = 2\,\mathrm{Tr}_1([\mathrm{Tr}_2(\hat{\rho}^2)\,\mathrm{Tr}_2(\hat{\rho})]\hat{H}_1) \tag{88}$$

where the indices 1 and 2 correspond to the two subsystems and we have assumed the standard two-particle extension of the (linear) Hamiltonian. Although we do not have much control over the behavior of the eigenvalues p_j of the reduced density matrix $\hat{\rho}_{(1)}$, we can infer that $\Sigma_j\, p_j$ is constant, whereas $\Sigma_j\, p_j^2$ is in general time dependent. Let us note that average energies of the two subsystems are separately conserved. This follows from the general property of the 3-bracket: For $H(\hat{\rho}) = H_1(\hat{\rho}_{(1)}) + H_2(\hat{\rho}_{(2)})$

$$\{H_1(\hat{\rho}_{(1)}), H(\hat{\rho}), S(\hat{\rho})\} = 0 \tag{89}$$

for any S (Czachor, 1997a). Therefore the probabilities p_j can be made time dependent without making the two subsystems interact with each other and

without changing energies of the subsystems, just by modifying the overall entropy of the composite system. So the change of entropy, say, by $C_2 \to C_2 + \varepsilon C_3$ at the global level, leads to the modification of the local subsystems. Such a modification will not occur if

$$[\mathrm{Tr}_2(\hat{\rho}^2), \mathrm{Tr}_2(\hat{\rho})] = 0 \tag{90}$$

which holds for a pure-state $\hat{\rho}$, or $\hat{\rho} = \hat{\rho}_{(1)} \otimes \hat{\rho}_{(2)}$. Still, strong correlations can also help since reduced density matrices occurring in a singlet state are proportional to unit matrices and the commutator vanishes. Systems described by entropies other than $C_2/2$ possess some kind of overall identity which is lost when it is physically meaningful to discuss their subsystems separately. This effect deserves a name. The fact that the subsystems "feel" that the total entropy (information) undegoes a change from C_2 to $C_2 + \varepsilon C_3$ although apparently "nothing happened" (no energy has been transfered between the neighboring subsystems) resembles the influence that Big Brother in G. Orwell's *1984* exerted on the inhabitants of Oceania "just by watching them."

It is not immediatly clear that the "Big Brother effect" is entirely unphysical. Its interpretation is obscured by our lack of understanding of the physical role played by the entropies C_n in the multiple-bracket scheme. It may be relevant to mention that C_2 is characteristic of the Rényi 2-entropy, which is the only α-entropy that characterizes a system whose gain of information is zero for all probability distributions. The analysis of this problem was given by Rényi (1960). Although this is the *first* paper where the notion of α-entropies was introduced, it does not seem to be known to the majority of experts in quantum mechanical information theory. The work typically quoted in the literature is Rényi (1961), where Rényi already departed from the natural definition of the information gain in favor of a "decrease of uncertainty." This latter modification was motivated by the problem with the vanishing gain for $\alpha = 2$.

A class of physical systems whose identity as a whole is associated with the way their entropy (or information) behaves are *living* organisms. Similarly, statistical properties of *societies* have dynamical properties strongly depending on information, and their dynamics cannot be regarded as a simple sum of individual activities. The fact that a possibility of gaining information can be formally related, via S, to nonlinearity of evolution resembles a similar phenomenon mentioned by Wigner (1967) in the context of the measurement problem ("paradox of a friend"). Whether such phenomena are in any way related to the 3-bracket dynamics is at the moment a matter of pure speculation.

11. QUANTIZATION OF CLASSICAL NAMBU DYNAMICS?

The $(2n + 1)$-bracket can be regarded as a nonlinear quantization of a classical $(n + 1)$-bracket with n classical Hamiltonian functions $H_1, \ldots,$

H_n. Indeed, the Liouville–von Neumann equation is characterized by *one* Hamiltonian operator $\hat{H}_1$, obtained by a quantization of a classical Hamiltonian function H_1. The requirement of linearity of evolution combined with the 3-bracket dynamics leads to the choice of $S = C_2/2$. Having n Hamiltonian functions H_k, we can obtain n Hamiltonian operators $\hat{H}_k$ after *some* quantization procedure (say, $p \to -i\hbar\nabla_x$, etc.). Representing the operators by kernels H_k^{ak}, $k = 1, \ldots, n$, we can consider the $(2n + 1)$-bracket equation

$$\dot{\rho}_a = z_{2n+1}\, \Omega_{aa_1 \ldots a_n b_1 \ldots b_n} H_1^{a_1} \ldots H_n^{a_n} \frac{\delta S_1}{\delta \rho_{b_1}} \ldots \frac{\delta S_n}{\delta \rho_{b_n}} \tag{91}$$

For $n > 1$ the equation is always nonlinear and its RHS vanishes on pure states. A self-adjoint Hilbert–Schmidt solution of the "quantized Nambu dynamics" may be interpreted as a density matrix because the spectrum of the solution is conserved. Had we started with the *linear* equation, which could be obtained by taking the $(n + 2)$-bracket

$$\dot{\rho}_a = z_{n+2} \Omega_{aa_1 \ldots a_n a_{n+1}} H_1^{a_1} \ldots H_n^{a_n} \rho^{a_{n+1}} \tag{92}$$

we would have obtained a dynamics which would not, in general, conserve $\mathrm{Tr}\,(\hat{\rho}^m)$ for $m > 1$ and there would be no guarantee that positivity of $\hat{\rho}$ is conserved.

This kind of nonlinear quantization differs from the procedure discussed in Takhtajan (1994), which was based on an n-bracket obtained by an antisymmetrization of a product of n oparators, or the Zariski product quantization proposed in Ditto *et al.* (1997). Also, all operator expressions involving an odd number of operator kernels, if described within our approach, must be excluded because the "metric" tensor used for the generalized structure constants would have to have an even number of indices, but such structure constants vanish (the 3-bracket involves antisymmetrization of two operators, the 5-bracket antisymmetrizes four operators, etc.). The quantization proposed originally by Nambu (1973) (cf. Garcia Sucre and Kálnay, 1975) is therefore also not equivalent to our formulation.

12. WHAT NEXT?

The formalism presented in this work is at a very preliminary stage of development. The main problem is how to solve the nonlinear density matrix equtions and how to extend the approach to a fully relativistic theory. Both questions are highly nontrivial. The equations of the form $i\dot{\rho} = [\hat{H}, \rho^n]$ bear some formal similarity to the Nahm equations studied in the $SU(2)$ monopole theory (Hitchin, 1983). Some recently developed techniques of solving matrix equations by a noncommutative version of a Darboux transform (Leble and

Zaitsev, 1997; Leble, 1997) may prove useful in this context. A natural candidate for a relativistic multiple-bracket formalism is the off-shell proper-time formulation. Some preliminary indefinite-metric results can be found in Czachor and Kuna (1997a), and work on positive-metric generalization of the Bargmann–Wigner off-shell equations (Czachor, 1997c) is in progress. The characteristic function off-shell approach developed recently by Naudts (1998) seems especially suited for this kind of generalization. A separate problem is the behavior of eigenvalues of reduced density matrices in the $\{\cdot\,,\cdot\}_{-H}$ scheme. These eigenvalues are in general time dependent and therefore the question of their interpretation is still unclear.

Note added in proof:

Explicit solutions of the $S = C_3/3$ Lie-Nambu equations, including an analysis of the "Big Brother effect", were obtained by the binary Darboux technique in S. B. Leble and M. Czachor, "Darboux-integrable nonlinear von Neumann equation", *Physical Review E* **58**, 7091(1998). A relationship of such equations to Tsallis-type nonextensive statistics was discussed in M. Czachor and J. Naudts, "Microscopic foundation of nonextensive statistcs", *Physical Review E* **59**. (March 1999)—in print.

ACKNOWLEDGMENTS

This work was written during my stays at ASI in Clausthal and CLEA in Brussels. Financial support from DAAD and the Polish–Flemish grant 007 are acknowledged. I am grateful to my friends I. Kanatchikov and J.-P. Ortega for their comments and suggestions.

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